Planar Binary Trees and Perturbative Calculus of Observables in Classical Field Theory

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Abstract

We study the Klein-Gordon equation coupled with an interaction term $(\Box+m^2)\phi+\lambda\psi^p$. For the linear Klein-Gordon equation, a kind of generalized Noether's theore gives us a conserved quantity. The purpose of this paper is to find an analogue of this conserved quantity in the interacting case. We see that we can do this perturbatively, and we define explicitely a conserved quantity, using a perturbative expansion based on Planar Binary Tree and a kind of Feynman rule. Only the case $p = 2$ is treated but our approach can be generalized to any ϕ^p–theory.

Introduction

In this paper, we study the Klein–Gordon equation coupled with a second order interaction term

$$(\Box + m^2)\varphi + \lambda\varphi^2 = 0 \qquad (E_\lambda)$$

where $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ is a scalar field and $\Box$ denote the operator $\frac{\partial^2}{\partial(x^0)^2} - \sum_{i=1}^n \frac{\partial^2}{\partial(x^i)^2}$. The constant m is a positive real number which is the mass and λ is a real parameter, the "coupling constant". Anyway our approach can be generalized to φ^k–theory for all $k \geq 3$.

For any $s \in \mathbb{R}$ we define the hypersurface $\Sigma_s \subset \mathbb{R}^{n+1}$ by $\Sigma_s := \{x = (x^0, \ldots, x^n) \in \mathbb{R}^{n+1}; x^0 = s\}$. The first variable x^0 plays the role of time variable, and so we will denote it by t. Hence we interpret Σ_s as a space–like surface by fixing the time to be equal to some constant s.

When λ equals zero, (E_λ) becomes the linear Klein–Gordon equation $(\Box + m^2)\varphi = 0$. Then it is well known (see e.g. [1]) that for any function ψ which satisfies $(\Box + m^2)\psi = 0$, if φ is a solution of (E_λ) for $\lambda = 0$ then for all $(s_1, s_2) \in \mathbb{R}^2$ we have

$$\int_{\Sigma_{s_1}} \left(\frac{\partial\psi}{\partial t}\varphi - \psi\frac{\partial\varphi}{\partial t}\right) \mathrm{d}\sigma = \int_{\Sigma_{s_2}} \left(\frac{\partial\psi}{\partial t}\varphi - \psi\frac{\partial\varphi}{\partial t}\right) \mathrm{d}\sigma \qquad (*)$$

This last identity can be seen as expressing the coincidence on the set of solutions φ of (E_0) of two functionals $\mathcal{I}_{\psi,s_1}$ and $\mathcal{I}_{\psi,s_2}$ where for all function $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ and all $s \in \mathbb{R}$,

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the functional $\mathcal{I}_{\psi,s}$ is defined by

$$\varphi \longmapsto \int_{\Sigma_s} \left(\frac{\partial \psi}{\partial t}\varphi - \psi \frac{\partial \varphi}{\partial t} \right) \mathrm{d}\sigma$$

So $(*)$ says exactly that on the set of solutions of the linear Klein-Gordon equation, the functional $\mathcal{I}_{\psi,s}$ does not depend on the time s.

This could be interpreted as a consequence of a generalized version of Noether's theorem, using the fact that the functional

$$\int_K \left(\frac{1}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{|\nabla \varphi|^2}{2} - \frac{m^2}{2}\varphi^2 \right) \mathrm{d}x$$

is infinitesimally invariant under the symmetry $\varphi \to \varphi + \varepsilon\chi$, where χ is a solution of (E_0), up to a boundary term.

This property is no longer true when $\lambda \neq 0$ i.e. when equation (E_λ) is not linear. The purpose of this article is to obtain a result analogous to $(*)$ in the nonlinear (interacting) case. Another way to formulate the problem could be, if we only know the field φ and its time derivative on a surface Σ_{s_1}, then how can we evaluate $\mathcal{I}_{\psi,s_2}$ where $s_2 \neq s_1$?

We will see that the computation of $\mathcal{I}_{\psi,s_2}$ can be done perturbatively when λ is small and s_2 is close to s_1. This perturbative computation takes the form of a power series over Planar Binary Trees. Note that Planar Binary Trees appear in other works on analogous Partial Differential Equations studied by perturbation (see [2], [3], [4], [5], [6], [7]) although the point of view differs with ours.

Let us express our main result. Without loss of generality we can suppose that $s_1 = 0$. Then using Planar Binary Trees and starting from a function ψ which satisfies $(\Box + m^2)\psi = 0$, we explicitly construct a family of functionals $(\Psi(b))_{b \in T(2)}$ indexed by the set $T(2)$ of Planar Binary Trees such that the following result holds

Theorem 1 *Let $q \in \mathbb{N}$ be such that $q > n/2$, $T > 0$ be a fixed time and $\psi \in \mathcal{C}^2([0,T], H^{-q})$ be such that $(\Box + m^2)\psi = 0$ in H^{-q}.*

i. For all φ in $\mathcal{C}^2([0,T], H^q)$ and $s \in [0,T]$ the power series in λ

$$\sum_{b \in T(2)} (-\lambda)^{|b|} \left\langle \Psi(b) \overleftrightarrow{\partial_s}^{\otimes \|b\|}, (\varphi, \ldots, \varphi) \right\rangle \qquad (\mathcal{S})$$

has a non zero radius of convergence R. More precisely we have

$$R \geq \left(4C_q M T \left[\|\varphi(s)\|_{H^q} + \|\frac{\partial \varphi}{\partial t}(s)\|_{H^q} \right] \right)^{-1}$$

here M and C_q are some constants.

ii. Let $\varphi \in \mathcal{E}$ be such that $(\Box + m^2)\varphi + \lambda\varphi^2 = 0$. If the condition

$$8M|\lambda|C_q T\|\varphi\|_{\mathcal{E}} \left(1 + |\lambda|C_q T\|\varphi\|_{\mathcal{E}}\right) < 1$$

is satisfied then the power series $(*)$ converges and we have for all $s \in [0, T]$

$$\sum_{b \in T(2)} (-\lambda)^{|b|} \left\langle \Psi(b) \overleftrightarrow{\partial_s}^{\otimes \|b\|}, (\varphi, \ldots, \varphi) \right\rangle = \int_{\Sigma_0} \left(\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t} \right)$$

The quantity $\|\varphi\|_{\mathcal{E}}$ can be evaluated using the initials conditions of φ. More details will be available in a upcoming paper [8]. This result can be generalized for ϕ^k–theory, $k \geq 3$, but instead of Planar Binary Trees we have to consider Planar k–Trees.

Beside the fact that the functional $\sum_b \lambda^{|b|} \Psi(b) \overleftrightarrow{\partial_s}^{\|b\|}$ provides us with a kind of generalized Noether's theorem charge, it can also help us to estimate the local values of the fields φ and $\frac{\partial \varphi}{\partial t}$. We just need to choose the test function ψ such that $\psi = 0$ on the surface Σ_0 and $\left. \frac{\partial \psi}{\partial t} \right|_{\Sigma_0}$ is an approximation of the Dirac mass at the point $x_0 \in \Sigma_0$. One gets the value of $\frac{\partial \varphi}{\partial t}$ at a point $x_0 \in \Sigma_0$ by exchanging ψ and $\frac{\partial \psi}{\partial t}$ in the previous reasoning.

Another motivation comes from the multisymplectic geometry. One of the purpose of this theory is to give a Hamiltonian formulation of the (classical) field theory similar to the symplectic formulation of the one dimensional Hamiltonian formalism (the Hamilton's formulation of Mechanics). If the time variable is replaced by several space-time variables, the multisymplectic formalism is based on an analogue to the cotangent bundle, a manifold equipped with a multisymplectic form similar to the symplectic form which appears naturally in the one dimensional theory. Then starting from the Lagrangian density which describes the dynamics of the field, one can construct a Hamiltonian function through a Legendre transform and obtain a geometric formulation of the problem. Note that this formalism differs from the standard Hamiltonian formulation of fields theory used by physicists (see e.g. [1]), in particular the multisymplectic approach is covariant i.e. compatible with the principles of special and general Relativity. For an introduction to the multisymplectic geometry one can refer to [9] and for more complete informations one can read the papers of F. Hélein and J. Kouneiher [10], [11].

The main motivation of the multisymplectic geometry is quantization, but it requires as preliminary to define the observable quantities, and the Poisson Bracket between these observables. A notion of observable have been defined by F. Hélein and J. Kouneiher. In the problem which interests us in this paper these observable quantities are essentially the functionals $\mathcal{I}_{\psi,s}$. In order to be able to compute the Poisson bracket between two such observables $\mathcal{I}_{\psi_1,s_1}$ and $\mathcal{I}_{\psi_2,s_2}$, we must be able to transport $\mathcal{I}_{\psi_1,s_1}$ into the surface Σ_{s_2}. When $\lambda = 0$, the identity $(*)$ gives us a way to do this manipulation, but when $\lambda \neq 0$ this is no longer the case. So F. Hélein proposed an approach based on perturbation; the reader will find more details on this subject in his paper [9].

In the first section, we begin the perturbative expansion by dealing with the linear case and the first order correction. The second section introduces the Planar Binary Trees which allow us to define the corrections of higher order, and the statement of the main result is given. Finally the last section contains the proof of the theorem.

1 Perturbative Calculus: Beginning Expansion

1.1 A simple case: $\lambda = 0$

Let us begin with the linear Klein-Gordon equation. Let $T > 0$ and $s \in [0, T]$ be a fixed positive time (the negative case is similar) and $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ a regular function. If φ belongs to S_0 i.e. be a solution of the linear Klein-Gordon equation, then

$$\int_{\Sigma_s} \left[\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t} \right] \mathrm{d}\sigma - \int_{\Sigma_0} \left[\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t} \right] \mathrm{d}\sigma = \int_D \left[\frac{\partial^2 \psi}{\partial t^2} \varphi - \psi \frac{\partial^2 \varphi}{\partial t^2} \right] \mathrm{d}x \tag{1.1}$$

where D denotes the set $D := [0, s] \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$. Since $\varphi \in S_0$ we have $\frac{\partial^2 \varphi}{\partial t^2} = \sum_i \frac{\partial^2 \varphi}{\partial z_i^2} - m^2 \varphi$ hence if one replaces $\frac{\partial^2 \varphi}{\partial t^2}$ in the right hand side of (1.1) and perform two integrations by parts, assuming that boundary terms vanish, one obtains

$$\int_{\Sigma_s} [\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t}] \mathrm{d}\sigma - \int_{\Sigma_0} [\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t}] \mathrm{d}\sigma = \int_D \varphi (\Box + m^2) \psi \tag{1.2}$$

Hence if we assume that ψ satisfies the linear Klein-Gordon equation $(\Box + m^2)\psi = 0$ then it follows that for all φ in S_0 and for all $s \in [0, T]$

$$\int_{\Sigma_s} \left[\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t} \right] \mathrm{d}\sigma - \int_{\Sigma_0} \left[\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t} \right] \mathrm{d}\sigma = 0 \tag{1.3}$$

We want to know how these computations are modified for $\varphi \in S_\lambda$ when $\lambda \neq 0$. For $\varphi \in S_\lambda$ we have $\frac{\partial^2 \varphi}{\partial t^2} = \sum_i \frac{\partial^2 \varphi}{\partial z_i^2} - m^2 \varphi - \lambda \varphi^2$. Hence instead of (1.3) one obtains

$$\int_{\Sigma_s} \left[\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t} \right] \mathrm{d}\sigma - \int_{\Sigma_0} \left[\frac{\partial \psi}{\partial t} \varphi - \psi \frac{\partial \varphi}{\partial t} \right] \mathrm{d}\sigma = \lambda \int_D \psi \varphi^2 \tag{1.4}$$

where ψ is supposed to satisfy the equation $(\Box + m^2)\psi = 0$. So the difference is no longer zero. However, one can remark that the difference seems[1] to be of order λ. This is the basic observation which leads to the perturbative calculus. One can look for another functional which annihilates the term $\lambda \int_D \varphi^2 \psi$.

1.2 First order correction: position of the problem

Let s be a non negative integer. In the previous section, it was shown that if one choose a function ψ such that $(\Box + m^2)\psi = 0$, then equality (1.4) occurs for all $\varphi \in S_\lambda$. The purpose of this section is to search for a counter–term of order λ which annihilates the right hand side of (1.4).

Let $\Psi^{(2)}$ be a smooth function $\Psi^{(2)} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ then for all $\varphi \in S_\lambda$ consider the quantity

$$\int_{\Sigma_s \times \Sigma_s} \Psi^{(2)} \left(\overleftarrow{\frac{\partial}{\partial t_1}} - \overrightarrow{\frac{\partial}{\partial t_1}} \right) \left(\overleftarrow{\frac{\partial}{\partial t_2}} - \overrightarrow{\frac{\partial}{\partial t_2}} \right) \varphi \otimes \varphi \, \mathrm{d}\sigma_1 \otimes \mathrm{d}\sigma_2 \tag{1.5}$$

[1]do not forget that the situation is actually more complicated because since φ satisfies the equation (E_λ), the field φ depends on λ.

We need to clarify the notation $\overleftarrow{A}$ and $\overrightarrow{B}$ for some given operator A and B. When the arrow is right to left (resp. left to right) the operator is acting on the left (resp. right).

If we assume that $\Psi^{(2)}$ satisfies the boundary condition

$$\forall\alpha\in\{0,1\}^2\ ,\ \left.\frac{\partial^{|\alpha|}\Psi^{(2)}}{\partial t^\alpha}\right|_{\Sigma_0\times\Sigma_s}=0 \tag{1.6}$$

then for all φ in S_λ we have

$$\int_{\Sigma_s\times\Sigma_s}\Psi^{(2)}\left(\overleftarrow{\frac{\partial}{\partial t_1}}-\overrightarrow{\frac{\partial}{\partial t_1}}\right)\left(\overleftarrow{\frac{\partial}{\partial t_2}}-\overrightarrow{\frac{\partial}{\partial t_2}}\right)\varphi\otimes\varphi$$
$$=\int_{D\times\Sigma_s}\frac{\partial}{\partial t_1}\left(\Psi^{(2)}\left(\overleftarrow{\frac{\partial}{\partial t_1}}-\overrightarrow{\frac{\partial}{\partial t_1}}\right)\left(\overleftarrow{\frac{\partial}{\partial t_2}}-\overrightarrow{\frac{\partial}{\partial t_2}}\right)\varphi\otimes\varphi\right)$$

here D denotes the set $D:=[0,s]\times\mathbb{R}^n$. Assume further that we have

$$\forall\alpha=(\alpha_1,\alpha_2)\in\{0,2\}\times\{0,1\}\quad;\quad\left.\frac{\partial^{|\alpha|}\Psi^{(2)}}{\partial t^\alpha}\right|_{D\times\Sigma_0}=0 \tag{1.7}$$

then we can do the same operation for the second variable t_2 and finally we get

$$\int_{\Sigma_s\times\Sigma_s}\Psi^{(2)}\left(\overleftarrow{\frac{\partial}{\partial t_1}}-\overrightarrow{\frac{\partial}{\partial t_1}}\right)\left(\overleftarrow{\frac{\partial}{\partial t_2}}-\overrightarrow{\frac{\partial}{\partial t_2}}\right)\varphi\otimes\varphi$$
$$=\int_{D\times D}\Psi^{(2)}\left(\overleftarrow{\frac{\partial^2}{\partial t_1^2}}-\overrightarrow{\frac{\partial^2}{\partial t_1^2}}\right)\left(\overleftarrow{\frac{\partial^2}{\partial t_2^2}}-\overrightarrow{\frac{\partial^2}{\partial t_2^2}}\right)\varphi\otimes\varphi$$

Now since φ belongs to S_λ we have $\frac{\partial^2\varphi}{\partial t^2}=\sum_i\frac{\partial^2\varphi}{\partial z_i^2}-m^2\varphi-\lambda\varphi^2$, hence one can replace the second derivatives with respect to time of φ and then perform integrations by parts in order to obtain : for all φ in S_λ the quantity (1.5) is given by

$$\int_{D\times D}\mathrm{d}x_1\mathrm{d}x_2\prod_{i=1}^{2}\big(\varphi(x_i)P_i+\lambda\varphi^2(x_i)\big)\Psi^{(2)}(x_1,x_2) \tag{1.8}$$

where P_i denotes the operator $P:=\square+m^2$ acting on the i–th variable. Here we assume that there are no boundary terms in the integrations by parts.

Using (1.8) and (1.4) one obtains that for all φ in S_λ we have

$$\Delta_\lambda=\lambda\Big[\int_{D\times D}\varphi^{\otimes 2}P_1P_2\Psi^{(2)}+\int_D\varphi^2\psi\Big]+\lambda^2\cdots$$

where Δ_λ denotes the quantity

$$\Delta_\lambda:=\int_{\Sigma_s}\left(\frac{\partial\psi}{\partial t}\varphi-\psi\frac{\partial\varphi}{\partial t}\right)\mathrm{d}\sigma+\lambda\int_{(\Sigma_s)^2}\Psi^{(2)}\left(\overleftarrow{\frac{\partial}{\partial t_1}}-\overrightarrow{\frac{\partial}{\partial t_1}}\right)\left(\overleftarrow{\frac{\partial}{\partial t_2}}-\overrightarrow{\frac{\partial}{\partial t_2}}\right)\varphi\otimes\varphi\,\mathrm{d}\sigma_1\otimes\mathrm{d}\sigma_2$$
$$-\int_{\Sigma_0}\left(\frac{\partial\psi}{\partial t}\varphi-\psi\frac{\partial\varphi}{\partial t}\right)\mathrm{d}\sigma \tag{1.9}$$

Hence if we choose a function $\Psi^{(2)}$ such that

$$P_1 P_2 \Psi^{(2)}(x_1, x_2) = -\delta(x_1 - x_2)\psi(x_1) \tag{1.10}$$

where δ is the Dirac operator then the first order term in the right hand side of (1.9) vanishes. But because of the hyperbolicity of the operator P, it seems difficult to control the regularity of such a function $\Psi^{(2)}$. Hence we need to allow $\Psi^{(2)}$ be in larger function space.

1.3 Function space background

Here we define the function spaces which allow us to express the correction terms. We fix some time $T > 0$.

Let $q \in \mathbb{Z}$ then we denote by $H^q(\mathbb{R}^n)$ (or simply H^q) the Sobolev space

$$H^q(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) \;\middle|\; (1 + |\xi|^2)^{q/2}\widehat{f}(\xi) \in L^2(R^n) \right\}$$

Then it is well known (see e.g. [12], [13], [14]) that H^q endowed with the norm $\|f\|_{H^q} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^q |\widehat{f}|^2(\xi)\mathrm{d}\xi$ is a Banach Space. Moreover one can see in every classical text book (see e.g. [14]) the following result

Proposition 1.1 *If $q > n/2$ then H^q is a Banach Algebra, i.e. there exists some constant $C_q > 0$ such that for all $(f, g) \in (H^q)^2$, $fg \in H^q$ and*

$$\|fg\|_{H^q} \leq C_q \|f\|_{H^q} \|g\|_{H^q}$$

Until now we fix some integer $q \in \mathbb{N}$ such that $q > n/2$.

Définition 1.1 *Let k be a positive integer, $k \in \mathbb{N}^*$. Then we denote by $\mathcal{E}^{k*}$ the space defined by*

$$\mathcal{E}^{k*} := \mathcal{C}^1([0, T]^k, \mathcal{L}_k(H^q))$$

where $\mathcal{L}_k(H^q)$ denotes the space of k–linear continuous forms over H^q, we will denote $\mathcal{E}^{1*}$ by $\mathcal{E}^*$.

Then $\mathcal{E}^{k*}$ together with the norm $\|\cdot\|_{k*}$ defined by

$$\|U\|_{k*} := \max_{\alpha \in \{0,1\}^k} \left(\sup_{\substack{t \in [0,T]^k \\ (f_1, \ldots, f_k) \in (H^q)^k \\ \|f_j\|_{H^q} \leq 1}} \left| \left\langle \frac{\partial^{|\alpha|} U}{\partial t^\alpha}(t), (f_1, \ldots, f_k) \right\rangle \right| \right)$$

is a Banach Space, here $\langle \cdot, \cdot \rangle$ denotes the duality brackets. For all $k \in \mathbb{N}^*$, we denote by $(\mathcal{E}^*)^{\otimes k}$ the space of elements U of $\mathcal{E}^{k*}$ such that there exists $(U_1, \ldots, U_k) \in \mathcal{E}^*$ such that $U = U_1 \otimes \cdots \otimes U_k$ i.e. for all $(f_1, \ldots, f_k) \in (H^q)^k$ and for all $t = (t_1, \ldots, t_k) \in [0, T]^k$

$$\langle U(t), (f_1, \ldots, f_k) \rangle = \langle U(t_1), f_1 \rangle \cdots \langle U(t_k), f_k \rangle$$

Then using the fact that the space of compact supported smooth functions is dense in H^q, one can easily prove the following property

Property 1.1 *for all k in $\mathbb{N}^*$, $(\mathcal{E}^*)^{\otimes k}$ is a dense subspace of $\mathcal{E}^{k*}$.*

We will denote by $\mathcal{E}$ the space defined by $\mathcal{E} := \mathcal{C}^2([0,T], H^q)$. Then $\mathcal{E}$ is a Banach space and we can see naturally $\mathcal{E}^{*k}$ as a subspace of $\mathcal{L}_k(\mathcal{E})$ the space of k–linear continuous form over $\mathcal{E}$; $\forall U \in \mathcal{E}^{k*}$ and $\forall \varphi = (\varphi_1, \ldots, \varphi_k) \in \mathcal{E}^k$

$$\langle U, \varphi \rangle := \int_0^T \mathrm{d}t_1 \cdots \int_0^T \mathrm{d}t_k \, \langle U(t_1, \ldots, t_k), (\varphi_1(t_1), \ldots, \varphi_k(t_k)) \rangle$$

Now let us generalize the expression (1.5) for the elements of $\mathcal{E}^{k*}$.

Définition 1.2 *Let U belong to $\mathcal{E}^*$ and $s \in [0,T]$, then we denote by $U \overleftrightarrow{\partial_s}$ the continuous linear form over $\mathcal{E}$ defined by $\forall \varphi \in \mathcal{E}$*

$$\left\langle U \overleftrightarrow{\partial_s}, \varphi \right\rangle := \left\langle \frac{\partial U}{\partial t}(s), \varphi(s) \right\rangle - \left\langle U(s), \frac{\partial \varphi}{\partial t}(s) \right\rangle \tag{1.11}$$

Then using the property 1.1 one can easily prove the following property

Property 1.2 *Let $k \in \mathbb{N}^*$ and $s \in [0,T]$ then there exists an unique operator $\overleftrightarrow{\partial_s}^{\otimes k}$: $\mathcal{E}^{*k} \longrightarrow \mathcal{L}_k(\mathcal{E})$ such that for all $U = U_1 \otimes \cdots \otimes U_k \in (\mathcal{E}^*)^{\otimes k}$ and for all $\varphi = (\varphi_1, \ldots, \varphi_k) \in \mathcal{E}^k$*

$$\left\langle \overleftrightarrow{\partial_s}^{\otimes k}(U), \varphi \right\rangle := \prod_{j=1}^{k} \left\langle U_j \overleftrightarrow{\partial_s}, \varphi_j \right\rangle$$

For $U \in \mathcal{E}^{*k}$ we will denote $\overleftrightarrow{\partial_s}^{\otimes k}(U)$ by $U \overleftrightarrow{\partial_s}^{\otimes k}$.

1.4 Resolution of the first order correction

Until now we fix some function $\psi \in \mathcal{C}^2([0,T], H^{-q})$ such that $(\Box + m^2)\psi = 0$. In this section we will define a functional $\Psi^{(2)}$ such that

$$\left\langle \psi \overleftrightarrow{\partial_s}, \varphi \right\rangle + \lambda \left\langle \Psi^{(2)} \overleftrightarrow{\partial_s}^{\otimes 2}, (\varphi, \varphi) \right\rangle - \left\langle \psi \overleftrightarrow{\partial_0}, \varphi \right\rangle \tag{1.12}$$

is of "order two respect with λ" for all $\varphi \in \mathcal{E}$ solution of (E_λ).

Définition 1.3 *Let $\overline{\Delta} : \mathcal{E}^* \longrightarrow \mathcal{E}^{*2}$ be the operator defined by $\forall t = (t_1, t_2) \in [0,T]^2$, $\forall (f_1, f_2) \in (H^q)^2$*

$$\langle \overline{\Delta} U(t_1, t_2), (f_1, f_2) \rangle := \int_0^T d\tau \, \langle U(\tau), (G(t_1) * f_1(\tau)) \, (G(t_2) * f_2(\tau)) \rangle$$

where for all $f \in H^q$, $t \in [0,T]$ and for all $\tau \in [0,T]$, $G(t) * f(\tau)$ denotes the element of H^q such that $\forall k \in \mathbb{R}^n$

$$\widehat{G(t) * f}(\tau)(k) := \theta(t-\tau) \frac{\sin((t-\tau)\omega_k)}{\omega_k} \widehat{f}(k) \tag{1.13}$$

where θ denote the Heavyside function[2] *and where $\omega_k := (m^2 + |k|^2)^{1/2}$.*

[2] $\theta(t) = 0$ if $t < 0$ and 1 otherwise.

Remark 1.1 *One can see $\overline{\Delta}U$ as a distribution $\overline{\Delta}U \in \mathcal{D}'((O,T)\times\mathbb{R}^n)$, and then one can show easily that we have the following formal expression for $\overline{\Delta}U$*

$$\overline{\Delta}U(x_1,x_2) = \int_{P_+} dy G_{ret}(x_1-y)G_{ret}(x_2-y)\psi(y) \tag{1.14}$$

where $P_+ = \{x\in\mathbb{R}^{n+1} | x^0>0\}$ and where $G_{ret}(z)$ denotes the retarded Green function of the Klein–Gordon operator

$$G_{ret}(z) := \frac{1}{(2\pi)^n}\theta(z^0)\int_{\mathbb{R}^n} d^n k \frac{\sin(z^0\omega_k)}{\omega_k} e^{ik.\overline{z}}$$

here $\overline{z}$ denotes the spatial part of $z\in\mathbb{R}^{n+1}$ i.e. $z=(z^0,\overline{z})$.

One can verify that $\overline{\Delta}$ is well defined and we have the following result

Proposition 1.2 *Let λ be a real number and $s\in[0,T]$ a fixed time. Let $\psi\in\mathcal{C}^2([0,T],H^{-q})$ be such that $\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m^2\psi = 0$. If $\varphi\in\mathcal{E}$ is a solution of the equation (E_λ) then the following inequality holds*

$$\begin{aligned}\Big|\left\langle \psi \overleftrightarrow{\partial_s}, \varphi\right\rangle + \lambda\left\langle \left(\overline{\Delta}\psi\right)\overleftrightarrow{\partial_s}^{\otimes 2}, (\varphi,\varphi)\right\rangle - \left\langle \psi\overleftrightarrow{\partial_0},\varphi\right\rangle\Big| & \\ \leq \lambda^2\left(\frac{s^2 C_q^2}{m}\|\varphi\|^3_{\mathcal{E}} + |\lambda|\frac{s^3C_q^3}{3m^2}\|\varphi\|^4_{\mathcal{E}}\right)\|\psi\|_{\infty,H^{-q}}\end{aligned} \tag{1.15}$$

This last proposition ensures that $\Psi^{(2)} := \overline{\Delta}\psi$ annihilates the term (1.4) of order one respect with λ.
<u>Proof</u>: (proposition 1.2)
Let $\psi\in\mathcal{C}^2([0,T],H^{-q})$ be such that $\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m^2\psi = 0$ and $\varphi\in\mathcal{E}$ be a solution of equation (E_λ). Then since ψ and φ are $\mathcal{C}^2$ the function $f: t\to \left\langle\psi\overleftrightarrow{\partial_t},\varphi\right\rangle$ admits derivative respect with t and $f'(t) = \left\langle \frac{\partial^2\psi}{\partial t^2}(t),\varphi(t)\right\rangle - \left\langle \psi(t),\frac{\partial^2\varphi}{\partial t^2}(t)\right\rangle$. But since ψ and φ satisfy $\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m^2\psi = 0$ and $\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi + m^2\varphi = -\lambda\varphi^2$ we have

$$f'(t) = \left\langle\psi(t),(\Delta-m^2)\varphi(t)\right\rangle - \left\langle\psi(t),\frac{\partial^2\varphi}{\partial t^2}(t)\right\rangle = \lambda\left\langle\psi(t),\varphi^2(t)\right\rangle$$

Hence we finally get $\forall s\in[0,T]$

$$\left\langle\psi\overleftrightarrow{\partial_s},\varphi\right\rangle - \left\langle\psi\overleftrightarrow{\partial_0},\varphi\right\rangle = f(s)-f(0) = \lambda\int_0^s \left\langle\psi(\tau),\varphi^2(\tau)\right\rangle \mathrm{d}\tau \tag{1.16}$$

and we recover the identity (1.4) of pages 4.

Now let us study the term of order one of the left hand side of (1.15). Using the definition 1.3 of $\overline{\Delta}$ one can show easily that it is given by the expression

$$\left\langle\left(\overline{\Delta}\psi\right)\overleftrightarrow{\partial_s}^{\otimes 2},(\varphi,\varphi)\right\rangle = \int_0^s \mathrm{d}\tau\int_{\mathbb{R}^n}\mathrm{d}k_1\int_{\mathbb{R}^n}\mathrm{d}k_2 M(s,\tau,k_1)M(s,\tau,k_2)\widehat{\psi}(\tau,k_1+k_2) \tag{1.17}$$

where $\forall(t,\tau)\in[0,T]^2$ and $\forall k\in\mathbb{R}^n$, the quantity $M(t,\tau,k)$ is given by

$$M(t,\tau,k) := \cos((t-\tau)\omega_k)\widehat{\overline{\varphi(t)}}(k) - \frac{\sin((t-\tau)\omega_k)}{\omega_k}\overline{\widehat{\frac{\partial\varphi(t)}{\partial t}}}(k) \tag{1.18}$$

The identity (1.17) can be seen as $\left\langle \left(\overline{\Delta}\psi\right) \overleftrightarrow{\partial_s}^{\otimes 2}, (\varphi,\varphi)\right\rangle = u(s)$ where $u:[0,T]\to\mathbb{R}$ is the continuous function given by

$$u(t) := \int_0^t \mathrm{d}\tau \int_{\mathbb{R}^n} \mathrm{d}k_1 \int_{\mathbb{R}^n} \mathrm{d}k_2 M(t,\tau,k_1)M(s,\tau,k_2)\widehat{\psi}(\tau,k_1+k_2)$$

Then in view of the definition (1.18) of $M(t,\tau,k)$ one can see that u admits derivative respect with t and since $u(0)=0$ we get $u(s)=\int_0^s u'(t)\mathrm{d}t$ which leads to

$$\begin{aligned}\left\langle \left(\overline{\Delta}\psi\right) \overleftrightarrow{\partial_s}^{\otimes 2}, (\varphi,\varphi)\right\rangle &= \int_0^s \mathrm{d}t \int_{(\mathbb{R}^n)^2} \mathrm{d}k_1\mathrm{d}k_2 \overline{\widehat{\varphi(t)}}(k_1)M(s,\tau,k_2)\widehat{\psi(t)}(k_1+k_2)\\ &- \int_0^s \mathrm{d}t\int_0^t \mathrm{d}\tau \int_{(\mathbb{R}^n)^2}\mathrm{d}k_1\mathrm{d}k_2 \frac{\sin((t-\tau)\omega_{k_1})}{\omega_{k_1}}\overline{\widehat{P\varphi(t)}}(k_1)M(s,\tau,k_2)\widehat{\psi(\tau)}(k_1+k_2) \quad (1.19)\end{aligned}$$

where P denotes the Klein–Gordon operator $P := \square + m^2$.

Then one can see the identity (1.19) as $\left\langle \left(\overline{\Delta}\psi\right) \overleftrightarrow{\partial_s}^{\otimes 2}, (\varphi,\varphi)\right\rangle = v(s)+w(s)$ where the functions $v,w:[0,T]\to\mathbb{R}$ are defined by

$$\begin{aligned} v(t) &= \int_0^t \mathrm{d}t_1 \int_{(\mathbb{R}^n)^2}\mathrm{d}k_1\mathrm{d}k_2 \overline{\widehat{\varphi(t_1)}}(k_1)M(t,\tau,k_2)\widehat{\psi(t_1)}(k_1+k_2)\\ w(t) &= -\int_0^t \mathrm{d}t_1 \int_0^{\min(t,t_1)}\mathrm{d}\tau \int_{(\mathbb{R}^n)^2}\mathrm{d}k_1\mathrm{d}k_2\\ &\qquad \frac{\sin((t_1-\tau)\omega_{k_1})}{\omega_{k_1}}\overline{\widehat{P\varphi(t_1)}}(k_1)M(t,\tau,k_2)\widehat{\psi(\tau)}(k_1+k_2)\end{aligned}$$

Then one can see that v and w admits derivative respect with t and that

$$\begin{aligned} v'(t) &= \int_{(\mathbb{R}^n)^2}\mathrm{d}k_1\mathrm{d}k_2\overline{\widehat{\varphi(t)}}(k_1)\overline{\widehat{\varphi(t)}}(k_2)\widehat{\psi(t)}(k_1+k_2)\\ &\quad - \int_0^t \mathrm{d}t_1 \int_{(\mathbb{R}^n)^2}\mathrm{d}k_1\mathrm{d}k_2\frac{\sin((t-t_1)\omega_{k_2})}{\omega_{k_2}}\overline{\widehat{\varphi(t_1)}}(k_1)\overline{\widehat{P\varphi(t)}}(k_2)\widehat{\psi(t_1)}(k_1+k_2)\end{aligned}$$

and

$$\begin{aligned} w'(t) &= \int_0^s \mathrm{d}t_1 \int_0^{\min(t_1,t)}\mathrm{d}\tau\int_{(\mathbb{R}^n)^2}\mathrm{d}k_1\mathrm{d}k_2\\ &\quad \frac{\sin((t_1-\tau)\omega_{k_1})}{\omega_{k_1}}\frac{\sin((t-\tau)\omega_{k_2})}{\omega_{k_2}}\overline{\widehat{P\varphi(t_1)}}(k_1)\overline{\widehat{P\varphi(t)}}(k_2)\widehat{\psi(\tau)}(k_1+k_2)\\ &\quad + \int_t^s \mathrm{d}t_1\int_{(\mathbb{R}^n)^2}\mathrm{d}k_1\mathrm{d}k_2\frac{\sin((t_1-t)\omega_{k_1})}{\omega_{k_1}}\overline{\widehat{P\varphi(t_1)}}(k_1)\overline{\widehat{\varphi(t)}}(k_2)\widehat{\psi(t_1)}(k_1+k_2)\end{aligned}$$

Hence since $v(0) = w(0) = 0$ and using the fact that $P\varphi(t) = -\lambda\varphi^2(t)$ and in view of (1.13) we finally get

$$
\begin{aligned}
&\Big\langle \big(\overline{\Delta}\psi\big)\, \overleftrightarrow{\partial_s}^{\otimes 2}, (\varphi,\varphi) \Big\rangle = \\
&\qquad \int_0^s \mathrm{d}\tau \left\langle \psi(\tau), \varphi^2(\tau) \right\rangle \\
&\qquad + 2\lambda \int_0^s \mathrm{d}t \int_0^s \mathrm{d}\tau \left\langle \psi(\tau), \big(G(t) * (\varphi^2(t))(\tau)\big)\, \varphi(\tau) \right\rangle \\
&\qquad + \lambda^2 \int_0^s \mathrm{d}t_1 \int_0^s \mathrm{d}t_2 \int_0^s \mathrm{d}\tau \left\langle \psi(\tau), \big(G(t_1) * (\varphi^2(t_1))(\tau)\big)\, \big(G(t_2) * (\varphi^2(t_2))(\tau)\big) \right\rangle
\end{aligned}
$$

Hence (1.16) and the last identity lead to

$$
\begin{aligned}
&\Big\langle \psi \overleftrightarrow{\partial_s}, \varphi \Big\rangle - \lambda \Big\langle \big(\overline{\Delta}\psi\big)\, \overleftrightarrow{\partial_s}^{\otimes 2}, (\varphi,\varphi) \Big\rangle - \Big\langle \psi \overleftrightarrow{\partial_0}, \varphi \Big\rangle = \\
&\qquad - 2\lambda^2 \int_0^s \mathrm{d}t \int_0^s \mathrm{d}\tau \left\langle \psi(\tau), \big(G(t) * (\varphi^2(t))(\tau)\big)\, \varphi(\tau) \right\rangle \\
&\qquad - \lambda^3 \int_0^s \mathrm{d}t_1 \int_0^s \mathrm{d}t_2 \int_0^s \mathrm{d}\tau \left\langle \psi(\tau), \big(G(t_1) * (\varphi^2(t_1))(\tau)\big)\, \big(G(t_2) * (\varphi^2(t_2))(\tau)\big) \right\rangle
\end{aligned}
\tag{1.20}
$$

Now to complete the proof it suffices to estimate the right hand side of (1.20). Using the definition (1.13) of $G(t) * f$ one can easily prove the following lemma

Lemma 1.4.1 *If f be in H^q then for all $(t,\tau) \in [0,T]^2$ we have $(G(t) * f)(\tau) \in H^q$ and $\|(G(t) * f)(\tau)\|_{H^q} \leq \frac{1}{m}\theta(t-\tau)\|f\|_{H^q}$.*

Hence using the lemma 1.4.1 and the property 1.1 one get

$$
\left| \int_0^s \mathrm{d}t \int_0^s \mathrm{d}\tau \left\langle \psi(\tau), \big(G(t) * (\varphi^2(t))(\tau)\big)\, \varphi(\tau) \right\rangle \right| \leq \frac{s^2 C_q^2}{2m} \|\psi\|_{\infty, H^{-q}} \|\varphi\|_{\mathcal{E}}^3
$$

and

$$
\left| \int_0^s \mathrm{d}t_1 \int_0^s \mathrm{d}t_2 \int_0^s \mathrm{d}\tau \left\langle \psi(\tau), \big(G(t_1) * (\varphi^2(t_1))(\tau)\big)\, \big(G(t_2) * (\varphi^2(t_2))(\tau)\big) \right\rangle \right| \leq \frac{s^3 C_q^3}{3m^2} \|\psi\|_{\infty, H^{-q}} \|\varphi\|_{\mathcal{E}}^4
$$

Then inserting these two inequality in (1.20) we finally get the result (1.15). ■

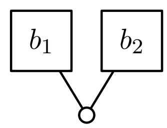
Hence we found a counter–term which annihilates the term (1.4) of order one respect with λ. But some extra new terms of high order have been introduced. Thus we need to find a functional $\lambda^2\Psi^{(3)}$ in order to delete the terms of order λ^2, and then an other functional $\lambda^3\Psi^{(4)}$ for those of order three etc. In order to picture all these extra terms, it will be suitable to introduce the following object: the *Planar Binary Tree*.

2 Planar Binary Tree

A *Planar Binary Tree* (PBT) is a connected oriented tree such that each vertex has either 0 or two sons. The vertices without sons are called the *leaves* and those with two sons are the *internal vertices*. For each Planar Binary Tree, There are an unique vertex which is the son of no other vertex, this vertex will be called the *root*. Since a Planar Binary Tree is oriented, one can define an order on the leaves. Until now we choose to arrange the leaves from left to right.

We will denote by $T(2)$ the set of Planar Binary Tree. Let denote by $|b|$ the number of internal vertices of a Planar Binary Tree b and $\|b\|$ the leave's number of b. Then one can easily show that we have $\|b\| = |b| + 1$. Let denote by ε the unique Planar Binary Tree with no internal vertex.

If b_1 and b_2 are two Planar Binary Trees, then we denote by $B_+(b_1, b_2)$ the Planar Binary Tree obtained by connecting a new root to b_1 on the left and to b_2 on the right.

$B_+(b_1, b_2) =$ 

Then one can easily show that $|B_+(b_1, b_2)| = |b_1| + |b_2| + 1$ and $\|B_+(b_1, b_2)\| = \|b_1\| + \|b_2\|$, and for all $b \in T(2)$, $b \neq \varepsilon$, there is an unique couple $(b_1, b_2) \in T(2)^2$ such that $b = B_+(b_1, b_2)$. For further details on the Planar Binary Trees, one can consult [15], [16], [17] or [18].

Définition 2.1 *We define inductively the family $(\overline{\Delta}(b))_{b\in T(2)}$ of functionals $\overline{\Delta}(b) : \mathcal{E}^* \longrightarrow \mathcal{E}^{*\|b\|}$ by*

$$\begin{cases} \overline{\Delta}(\varepsilon) := id \\ \forall (b_1, b_2) \in T(2)^2 \ ; \overline{\Delta}(B_+(b_1, b_2)) := (\overline{\Delta}(b_1) \otimes \overline{\Delta}(b_2)) \circ \overline{\Delta} \end{cases} \tag{2.1}$$

where for $\mathcal{U} : \mathcal{E}^* \to \mathcal{E}^{*k}$ and $\mathcal{V} : \mathcal{E}^* \to \mathcal{E}^{*l}$, $\mathcal{U} \otimes \mathcal{V}$ denotes the unique functional from $\mathcal{E}^{*2}$ to $\mathcal{E}^{*(k+l)}$ such that for all $U = U_1 \otimes U_2 \in (\mathcal{E}^*)^{\otimes 2}$, $\mathcal{U} \otimes \mathcal{V}(U) = \mathcal{U}(U_1) \otimes \mathcal{V}(U_2)$.

Let ψ belong to $\mathcal{E}^*$, then we consider the family $(\Psi(b))_{b\in T(2)}$ defined by

$$\Psi(b) := \overline{\Delta}(b)(\psi) \in \mathcal{E}^{*\|b\|}$$

Then using remark 1.1 we can see that formally the functionals $\Psi(b)$, $b \in T(2)$, can be constructed using the following rules :

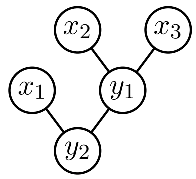
1. attach to each leaf of b the space–time variable $x_1, x_2, \ldots, x_{\|b\|}$ with respect to the order of the leaves.

2. for each internal vertex attach a space–time integration variable $y_i \in \mathbb{R}^{n+1}$ and integrate this variable over P_+.

3. for each line between the vertices v and w where the depth of v is lower than the w's, put a factor $G_{ret}(a_v - a_w)$ where a_v (resp. a_w) is the space–time variable associated with v (resp. w).

4. finally multiply by $\psi(a_r)$ where a_r is the space–time variable attached to the root of the Planar Binary Tree b.

To fix the ideas, let us treat an example. Let $b \in T_3$ be the Planar Binary Tree described by the following graph

$b =$ 

Then using definition 2.1 we have $\Psi(b) = (id \otimes \overline{\Delta}) \circ \overline{\Delta}\psi$ and formally for $x = (x_1, x_2, x_3) \in ([0,T] \times \mathbb{R}^n)^3$, $\Psi(b)(x)$ is given by the following

$$\Psi(b)(x) = \iint_{P_+} \mathrm{d}y_1 \mathrm{d}y_2 G_{ret}(x_1 - y_2) G_{ret}(y_1 - y_2) G_{ret}(x_2 - y_1) G_{ret}(x_3 - y_1) \psi(y_2)$$

Theorem 2.1 *i. Let $\psi \in \mathcal{C}^2([0,T], H^{-q})$ be such that $(\Box + m^2)\psi = 0$ in H^{-q}. Let φ be in $\mathcal{E}$ and $s \in [0,T]$, then the power series in λ*

$$\sum_{b \in T(2)} (-\lambda)^{|b|} \left\langle \Psi(b) \overleftrightarrow{\partial_s}^{\otimes \|b\|}, (\varphi, \ldots, \varphi) \right\rangle \qquad (*)$$

has a non zero radius of convergence R. More precisely we have

$$R \geq \left(4 C_q M T \left[\|\varphi(s)\|_{H^q} + \|\frac{\partial \varphi}{\partial t}(s)\|_{H^q} \right] \right)^{-1}$$

here M is defined by $M := \max(\frac{1}{m}, 1)$ and C_q is the constant of the property 1.1.

ii. Let $\varphi \in \mathcal{E}$ be such that $(\Box + m^2)\varphi + \lambda\varphi^2 = 0$. If the condition

$$8M|\lambda| C_q T \|\varphi\|_{\mathcal{E}} \left(1 + |\lambda| C_q T \|\varphi\|_{\mathcal{E}}\right) < 1 \qquad (2.2)$$

is satisfied then the power series $(*)$ converges and we have for all $s \in [0,T]$

$$\sum_{b \in T(2)} (-\lambda)^{|b|} \left\langle \Psi(b) \overleftrightarrow{\partial_s}^{\otimes \|b\|}, (\varphi, \ldots, \varphi) \right\rangle = \left\langle \psi \overleftrightarrow{\partial_0}, \varphi \right\rangle$$

Remark: Note that it is possible to control the norm $\|\varphi\|_{\mathcal{E}}$ with the norm of initials data. More precisely for all any $(\varphi^0, \varphi^1) \in (H^q)^2$, $\lambda \in \mathbb{R}$ and $T \in \mathbb{R}$ such that $T|\lambda| \|(\varphi^0, \varphi^1)\|$ is small enough, it is possible to construct a solution $\varphi \in \mathcal{C}^2([0,T], H^q)$ of (E_λ) such that $\varphi(0,\cdot) = \varphi^0$ and $\frac{\partial \varphi}{\partial t}(0,\cdot) = \varphi^1$. Then one can control $\|\varphi\|_{\mathcal{E}}$ using $\|(\varphi^0, \varphi^1)\|$. A proof of this result, based on a remark of Christian Brouder ([7]) will be expand in [8].

Let us comment this last proposition. First of all, using definition (1.2) of $\overleftrightarrow{\partial_s}$, one can remark that the power series $(*)$ depends only on $\varphi(s,\cdot)$ and $\frac{\partial\varphi}{\partial t}(s,\cdot)$. Hence the theorem answers the original question.

We have written the solution for s non negative, but the study can be done in the same way for negative s. Finally the result exposed in the proposition 2.1 can be generalized to ϕ^p–theory i.e. for the equation $(\Box + m^2)\varphi + \lambda\varphi^p = 0$, $p \geq 2$. But the set of Planar Binary Trees must be replaced by T(p), the set of *Planar p-Trees* i.e. oriented rooted trees which vertices have 0 or p sons, then the definition of $(\Psi(b))_{b\in T(p)}$ remains the same i.e. $\Psi(b) := \overline{\Delta}^{(p)}(b)\psi$ where $\overline{\Delta}^{(p)}(b)$ is an adaptation of definition 2.1 for p–trees. Then an analogue of theorem 2.1 holds but the condition (2.2) must be adapted.

3 Proof of the main proposition

3.1 Radius of convergence

Let us deal with the first part of theorem 2.1. First we will prove the following lemma

Lemma 3.1.1 *Let ψ belong to $\mathcal{E}^*$ and $b \in T(2)$, then we have*

$$\|\overline{\Delta}(b)\psi\|_{*\|b\|} \leq (C_q MT)^{|b|}\,\|\psi\|_{*1} \tag{3.1}$$

where M and C_q are the constants which appear in theorem (2.1).

Proof: (lemma 3.1.1)
We will show (3.1) inductively with respect to $|b|$ the number of internal vertices of b.

If $b = \varepsilon$ then inequality (3.1) is satisfied. Let $N \in \mathbb{N}$, suppose that (3.1) is true for all $b \in T(2)$ such that $|b| \leq N$ and let $b \in T(2)$ be such that $|b| = N + 1 \geq 1$. Then b writes $b = B_+(b_1, b_2)$ and by definition we have $\Psi(b) = \overline{\Delta}(b)\psi = (\overline{\Delta}(b_1) \otimes \overline{\Delta}(b_2)) \circ \overline{\Delta}\psi$. Since $(\mathcal{E}^*)^{\otimes 2}$ is dense in $\mathcal{E}^{*2}$ there is a sequence $U_n = U_n^{(1)} \otimes U_n^{(2)} \in (\mathcal{E}^*)^{\otimes 2}$, $n \in \mathbb{N}$, such that $U_n \to \overline{\Delta}\psi$ in $\mathcal{E}^{*2}$. Then one can show easily that

$$\|\overline{\Delta}(b_1) \otimes \overline{\Delta}(b_2))U_n\|_{*(\|B_+(b_1,b_2)\|)} = \|\overline{\Delta}(b_1)U_n^{(1)}\|_{\|b_1\|}\|\overline{\Delta}(b_2)U_n^{(2)}\|$$

but since $|B_+(b_1, b_2)| = |b_1| + |b_2| + 1$ we have $|b_1| \leq N$ and $|b_2| \leq N$, hence (3.1) is valid for b_1 and b_2 we finally get

$$\begin{aligned}\|\overline{\Delta}(b_1) \otimes \overline{\Delta}(b_2))U_n\|_{*(\|B_+(b_1,b_2)\|)} &\leq (C_q MT)^{|b_1|+|b_2|}\,\|U_n^{(1)}\|_{*1}\|U_n^{(2)}\|_{*1}\\ &= (C_q MT)^{|b_1|+|b_2|}\,\|U_n^{(1)} \otimes U_n^{(2)}\|_{*2}\end{aligned}$$

Then taking the limit $n \to \infty$ in the previous inequality leads to

$$\|\overline{\Delta}(B_+(b_1, b_2))\psi\|_{*(\|B_+(b_1,b_2)\|)} \leq (C_q MT)^{|b_1|+|b_2|}\,\|\overline{\Delta}\psi\|_{2*} \tag{3.2}$$

Let f_1 and f_2 belong to H^q then by definition we have for all $(t_1, t_2) \in [0, T]^2$ and $\alpha = (\alpha_1, \alpha_2) \in \{0, 1\}^2$

$$\left\langle \frac{\partial^{|\alpha|}\overline{\Delta}\psi}{\overline{\Delta}t^\alpha}(t_1, t_2), (f_1, f_2) \right\rangle = \int_0^T \mathrm{d}\tau\,\langle\psi(\tau), (G^{\alpha_1}(t_1)f_1(\tau))\,(G^{\alpha_2}(t_2)f_2(\tau))\rangle \tag{3.3}$$

where $G^0(t)f(\tau) := G(t)f(\tau)$ has been defined in the section 1.4 page 7 and where $G^1(t)f(\tau) \in H^q$ is the function such that $\forall \overrightarrow{k} \in \mathbb{R}^n$

$$\widehat{G^1(t_1)f}(\tau)(\overrightarrow{k}) := \theta(t_1-\tau)\cos((t_1-\tau)\omega_{\overrightarrow{k}})\widehat{f}(k)$$

Then one can easily show that $\|G^1(t)f(\tau)\|_{H^q} \leq \frac{1}{m}\theta(t-\tau)\|f\|_{H^q}$ and $\|G^1(t)f(\tau)\|_{H^q} \leq \theta(t-\tau)\|f\|_{H^q}$. Hence inserting these results in (3.3) and using property 1.1 we get

$$\|\overline{\Delta}\psi\|_{2*} \leq MC_qT\|\psi\|_{*1}$$

So in view of (3.2) we see that the estimation (3.1) is valid for $b = B_+(b_1, b_2)$. ■

Let φ belong to $\mathcal{E}$ then lemma 3.1.1 shows that for all $s \in [0,T]$ and for all $b \in T(2)$ we have

$$\left|\left\langle \Psi(b)\overleftrightarrow{\partial_s}, (\varphi,\ldots,\varphi)\right\rangle\right| \leq (C_qMT)^{|b|}\,\|\psi\|_{*1}\left[\|\varphi(s)\|_{H^q} + \|\frac{\partial\varphi}{\partial t}(s)\|_{H^q}\right]^{\|b\|}$$

then using the fact (see e.g. [16]) that the number p_N of Planar Binary Tree b such that $|b| = N$ satisfies $p_N \leq 4^N$ we finally get the first part of theorem 2.1, i.e. the power series in λ defined by $\sum_{b\in T(2)}(-\lambda)^{|b|}\left\langle \Psi(b)\overleftrightarrow{\partial_s}, (\varphi,\ldots,\varphi)\right\rangle$ has a non–zero radius of convergence R and

$$R \geq \left(4C_qMT\left[\|\varphi(s)\|_{H^q} + \|\frac{\partial\varphi}{\partial t}(s)\|_{H^q}\right]\right)^{-1} > 0$$

3.2 Algebraic calculations

Let us fix some time s in $[0,T]$, then we define the operator $P : \mathcal{E}^* \longrightarrow \mathcal{F}'$ where $\mathcal{F} \subset \mathcal{E}$ denotes the space $\mathcal{F} := \mathcal{C}^2([0,T],H^q)\cap\mathcal{C}^0([0,T],H^{q+2})$ by for all $U \in \mathcal{E}^*$ and for all $\varphi \in \mathcal{F}$

$$\langle PU, \varphi\rangle := \left\langle U\overleftrightarrow{\partial_s}, \varphi\right\rangle - \left\langle U\overleftrightarrow{\partial_0}, \varphi\right\rangle + \int \mathrm{d}\tau\,\langle U(\tau), (\Box + m^2)\varphi(\tau)\rangle \tag{3.4}$$

here $\Box + m^2$ denotes the operator $\mathcal{F} \to \mathcal{E}$ defined by $\Box = \frac{\partial^2}{\partial t^2} - \Delta$. Let k be an integer $k \in \mathbb{N}^2$ then for all $I \subset [\![1,k]\!]$ we denote by P_I^k the unique continuous operator $P_I^k : \mathcal{E}^{k*} \longrightarrow \mathcal{L}_k(\mathcal{F})$ such that for all $U = U_1 \otimes \cdots U_k \in (\mathcal{E}^*)^{\otimes k}$ and for all $\varphi = (\varphi_1,\ldots,\varphi_k) \in \mathcal{F}^k$

$$\left\langle P_I^k U, \varphi\right\rangle = \prod_{i\in I}\langle PU_i, \varphi_i\rangle \prod_{j\notin I}\int_0^T \langle U_j(\tau_j), \varphi_j(\tau_j)\rangle\,\mathrm{d}\tau_j$$

Let $\varphi \in \mathcal{E}$ be a solution of (E_λ) i.e. such that $(\Box + m^2)\varphi = -\lambda\varphi^2$. Then in view property 1.1 φ belongs to $\mathcal{F}$. Let b be a Planar Binary Tree such that $b \neq \varepsilon$ and denote by k the leave's number of b, $k := \|b\|$. Then in view of definition 2.1 one can easily see that for all $J \subset [\![1,k]\!]$ we have $\Psi(b)\overleftrightarrow{\partial_0^I} = 0$ hence the definition (3.4) of P leads to

$$\left\langle \Psi(b)\overleftrightarrow{\partial_s}^{\otimes k}, (\varphi,\ldots,\varphi)\right\rangle = \sum_{I\subset[\![1,k]\!]}\lambda^{k-|I|}\left\langle P_I^k\Psi(b), (\varphi^{\alpha_1^I},\ldots,\varphi^{\alpha_k^I})\right\rangle \tag{3.5}$$

where $\alpha_j^I = 2$ if $j \notin I$ and $\alpha_j^I = 1$ otherwise; here we use the fact that φ satisfies $-(\Box + m^2)\varphi = \lambda\varphi^2$. Moreover the proof of proposition 1.2 shows that if one choose $\psi \in \mathcal{C}^2([0,T], H^{-q})$ such that $(\Box + m^2)\psi = 0$ then we have

$$\left\langle \psi \overleftrightarrow{\partial_s}, \varphi \right\rangle - \left\langle \psi \overleftrightarrow{\partial_0}, \varphi \right\rangle = -\lambda \left\langle \psi, \varphi^2 \right\rangle = -\lambda \int_0^s \left\langle \psi(\tau), \varphi^2(\tau) \right\rangle \mathrm{d}\tau \tag{3.6}$$

i.e. $P\psi = 0$. For $N \in \mathbb{N}^*$ let denote by Δ_N the finite sum

$$\Delta_N := \sum_{\substack{b \in T(2) \\ |b| \leq N}} (-\lambda)^{|b|} \left\langle \Psi(b) \overleftrightarrow{\partial_s}^{\otimes \|b\|}, (\varphi, \ldots, \varphi) \right\rangle - \left\langle \psi \overleftrightarrow{\partial_0}, \varphi \right\rangle$$

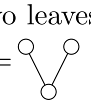
Then (3.6) and (3.5) lead to

$$\Delta_N = \sum_{\beta=1}^{2N-1} \lambda^{\beta-1} \sum_{\substack{1 \leq k \leq N \\ 0 \leq l \leq k \\ k+l=\beta}} \sum_{\substack{b \in T(2) \\ \|b\|=k}} \sum_{\substack{I \subset [\![1,k]\!] \\ |I|=k-l}} (-1)^{|b|} \left\langle P_I^k \Psi(b), (\varphi^{\alpha_1^I}, \ldots, \varphi^{\alpha_k^I}) \right\rangle \tag{3.7}$$

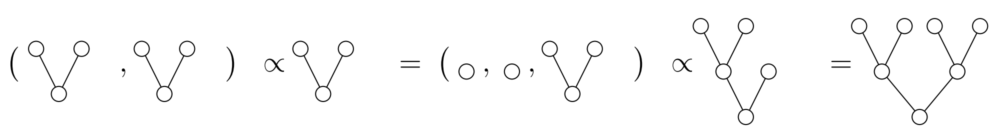
Let $\beta \in \mathbb{N}^*$ be such that $\beta \leq N$ then Δ_N^β the term of order β with respect to λ in (3.7) writes

$$\begin{aligned} \Delta_N^\beta = & \sum_{\substack{b \in T(2) \\ \|b\|=\beta}} (-1)^{|b|} \left\langle P_{[\![1,\beta]\!]}^\beta \Psi(b), (\varphi, \ldots, \varphi) \right\rangle \\ & + \sum_{\substack{1 \leq l \leq k \leq \beta \\ k+l=\beta}} \sum_{\substack{a \in T(2) \\ \|a\|=k}} \sum_{\substack{I \subset [\![1,k]\!] \\ |I|=k-l}} (-1)^{|a|} \left\langle P_I^k \Psi(a), (\varphi^{\alpha_1^I}, \ldots, \varphi^{\alpha_k^I}) \right\rangle \end{aligned} \tag{3.8}$$

Let us focus on the first sum of this last identity. We need some extra structure on the set of Planar Binary Tree. Two special Planar Binary Trees play an important role : the Planar Binary Tree ε with one leaf, and Y the one with two leaves

$\varepsilon = \ \circ$ and $Y =$ 

We also define the *growing* operation. Let b be a Planar Binary Tree with k leaves and $E = (E_1, \ldots, E_k)$ be a k-uplet in $\{\varepsilon, Y\}^k$. We call the *growing of E on b* and denote by $E \propto b$ the Planar Binary Tree obtained by replacing the i-th leaf of b by E_i. As an example



For $E \in \{\varepsilon, Y\}^k$ we denote by $n_Y(E)$ the occurrence number of Y in E i.e. $n_Y(E) := \mathrm{Card}\{i | E_i = Y\}$. Then we have the combinatorial lemma

Lemma 3.2.1 *1. Let b be a Planar Binary Tree with β leaves, $\beta \geq 2$. Then we have*

$$-\sum_{\substack{1\leq l\leq k\leq\beta\\ k+l=\beta}} \sum_{\substack{a\in T_k\\ E\in\{\varepsilon,Y\}^k|n_Y(E)=l\\ \text{such that } E\propto a=b}} (-1)^{|a|} = (-1)^{|b|}$$

2. Let $p \in \mathbb{N}^*$, $a \in T(2)$ be such that $\|a\| = p$ and $E \in \{\varepsilon, Y\}^p$ then we have $\|E \propto a\| = p + n_Y(E)$ and

$$\left\langle P^{p+n_Y(E)}_{[\![1,p+n_Y(E)]\!]}\Psi(E\propto a),(\varphi,\ldots,\varphi)\right\rangle = \left\langle P^p_{I_E}\Psi(a),(\varphi^{\alpha_1^{I_E}},\ldots,\varphi^{\alpha_p^{I_E}}\right\rangle$$

where $I_E := \{j \in [\![1,p]\!]$ such that $E_j = \varepsilon\}$ and $\alpha_j^{I_E} = 2$ if $j \notin I_E$ and 1 otherwise.

We postpone the proof until appendix. Then the point 1 lemma 3.2.1 leads to

$$\sum_{\substack{b\in T(2)\\ \|b\|=\beta}} (-1)^{|b|}\left\langle P^\beta_{[\![1,\beta]\!]}\Psi(b),(\varphi,\ldots,\varphi)\right\rangle$$

$$= -\sum_{\substack{1\leq l\leq k\leq\beta\\ k+l=\beta}} \sum_{\substack{a\in T_k\\ E\in\{\varepsilon,Y\}^k|n_Y(E)=l}} (-1)^{|a|}\left\langle P^\beta_{[\![1,\beta]\!]}\Psi(E\propto a),(\varphi,\ldots,\varphi)\right\rangle \quad (3.9)$$

But since $E \in \{\varepsilon, Y\}^p$ is entirely determined by p and I_E, the point 2 of lemma 3.2.1 and identity (3.9) lead to

$$\sum_{\substack{b\in T(2)\\ \|b\|=\beta}} (-1)^{|b|}\left\langle P^\beta_{[\![1,\beta]\!]}\Psi(b),(\varphi,\ldots,\varphi)\right\rangle$$

$$= -\sum_{\substack{1\leq l\leq k\leq\beta\\ k+l=\beta}} \sum_{\substack{a\in T(2)\\ \|a\|=k}} \sum_{\substack{I\subset[\![1,k]\!]\\ |I|=k-l}} (-1)^{|a|}\left\langle P^k_I\Psi(a),(\varphi^{\alpha_1^I},\ldots,\varphi^{\alpha_k^I})\right\rangle$$

then inserting this last identity in (3.8) we finally get that for all $\beta \leq N$, $\Delta^N_\beta = 0$.

Hence we have shown that $\Delta_N = \sum_{\beta=1}^{2N-1}\lambda^{\beta-1}\Delta^N_\beta$ is of order N with respect to λ. To complete the proof it suffices to show that Δ_N converges to 0 when N tends to infinity.

3.3 analytic study

We have shown that all the terms of order β with $\beta \leq N$ in identity (3.7) vanish, hence we have

$$\Delta_N = \sum_{\beta=N+1}^{2N-1}\lambda^{\beta-1} \sum_{\substack{1\leq l\leq k\leq N\\ k+l=\beta}} \sum_{\substack{b\in T(2)\\ \|b\|=k}} \sum_{\substack{I\subset[\![1,k]\!]\\ |I|=k-l}} (-1)^{|b|}\left\langle P^k_I\Psi(b),(\varphi^{\alpha_1^I},\ldots,\varphi^{\alpha_k^I})\right\rangle \quad (3.10)$$

We have to estimate the right hand side of this last identity. Let us prove the following lemma

Lemma 3.3.1 *Let $k \in \mathbb{N}^*$, $k \geq 2$ and $b \in T(2)$ be such that $\|b\| = k$, then for all $I \subset [\![1,k]\!]$, $\varphi \in \mathcal{E}$ solution of (E_λ) and $\psi \in \mathcal{E}^*$ we have*

$$\left|\left\langle P_I^k \Psi(b), (\varphi^{\alpha_1^I}, \ldots, \varphi^{\alpha_k^I})\right\rangle\right| \leq \\ (C_q T)^{k-|I|} \|\varphi\|_{\mathcal{E}}^{2k-|I|} (C_q M T)^{k-1} \left(2 + |\lambda| C_q T \|\varphi\|_{\mathcal{E}}\right)^{|I|} \|\psi\|_{\mathcal{E}^*} \quad (3.11)$$

Proof: (lemma 3.3.1)
Let $\tilde{P}$ denote the operator $\tilde{P} : \mathcal{E}^* \longrightarrow \mathcal{F}'$ defined by for all $U \in \mathcal{E}^*$ and for all $\varphi \in \mathcal{F}$

$$\left\langle \tilde{P}U, \varphi \right\rangle := \left\langle U \overleftrightarrow{\partial_s}, \varphi \right\rangle + \int \mathrm{d}\tau \left\langle U(\tau), (\Box + m^2)\varphi(\tau)\right\rangle \quad (3.12)$$

and for $k \in \mathbb{N}^*$ and $I \subset [\![1,k]\!]$ we denote by $\tilde{P}_I^k$ the unique continuous operator $\tilde{P}_I^k : \mathcal{E}^{k*} \longrightarrow \mathcal{L}_k(\mathcal{F})$ such that for all $U = U_1 \otimes \cdots U_k \in (\mathcal{E}^*)^{\otimes k}$ and for all $\varphi = (\varphi_1, \ldots, \varphi_k) \in \mathcal{F}^k$

$$\left\langle \tilde{P}_I^k U, \varphi \right\rangle = \prod_{i \in I} \left\langle \tilde{P} U_i, \varphi_i \right\rangle \prod_{j \notin I} \int_0^T \langle U_j(\tau_j), \varphi_j(\tau_j)\rangle \, \mathrm{d}\tau_j$$

Then since $\forall b \in T(2)$, $b \neq \varepsilon$ and $\forall \alpha \in \{0,1\}^{\|b\|}$ the operator $\frac{\partial^\alpha \Psi(b)}{\partial t^\alpha}$ satisfies $\frac{\partial^\alpha \Psi(b)}{\partial t^\alpha}(t) = 0$ for all $t \in \cup_{j=0}^{\|b\|-1} [0,T]^j \times \{0\} \times [0,T]^{\|b\|-1-j}$ we have $P_I^{\|b\|}\Psi(b) = \tilde{P}_I^{\|b\|}\Psi(b)$ for all $b \in T(2)$, $b \neq \varepsilon$ and for all $I \subset [\![1, \|b\|]\!]$.

Let $\varphi \in \mathcal{E}$ be a solution of (E_λ) and $U \in \mathcal{E}^*$ then $(\Box + m^2)\varphi = -\lambda\varphi^2$ and using property 1.1 and definition (3.12) we get

$$\begin{aligned} \left|\left\langle \tilde{P}U, \varphi \right\rangle\right| &\leq \|U\|_{*1} \left(\|\varphi(s)\|_{H^q} + \|\frac{\partial \varphi}{\partial t}(s)\|_{H^q} \right) + C_q |\lambda| T \|U\|_{*1} \|\varphi\|_{\mathcal{E}}^2 \\ &\leq \left(2 + C_q |\lambda| T \|\varphi\|_{\mathcal{E}}\right) \|\varphi\|_{\mathcal{E}} \|U\|_{*1} \end{aligned} \quad (3.13)$$

Let $b \in T(2)$, $\|b\| =: k \geq 2$, then consider a sequence $U_n = (U_n^{(1)} \otimes \cdots \otimes U_n^{(k)}) \in (\mathcal{E}^*)^{\otimes k}$ such that $U_n \to \Psi(b)$ in $\mathcal{E}^{*k}$. Let $\varphi \in \mathcal{E}$ be a solution of (E_λ) and $I \subset [\![1,k]\!]$ then using (3.13) and property 1.1 we get

$$\left|\left\langle \tilde{P}_I^k U_n, \varphi \right\rangle\right| \leq \left(2 + C_q |\lambda| T \|\varphi\|_{\mathcal{E}}\right)^{|I|} \|\varphi\|_{\mathcal{E}}^{|I|} (T C_q)^{k-|I|} \|\varphi\|_{\mathcal{E}}^{2k-|I|} \|U_n\|_{*k}$$

then taking the limit $n \to \infty$ in the last inequality we get

$$\left|\left\langle \tilde{P}_I^k \Psi(b), \varphi \right\rangle\right| \leq \left(2 + C_q |\lambda| T \|\varphi\|_{\mathcal{E}}\right)^{|I|} \|\varphi\|_{\mathcal{E}}^{|I|} (T C_q)^{k-|I|} \|\varphi\|_{\mathcal{E}}^{2k-|I|} \|\Psi(b)\|_{*k}$$

hence lemma 3.1.1 completes the proof of lemma 3.3.1. ■

Then using (3.10), lemma 3.3.1 and the fact that the number p_k of Planar Binary Tree b such that $|b| = k$ satisfies $p_k \leq 4^k$ we get

$$|\Delta_N| \leq \\ \|\psi\|_{\mathcal{E}^*} \sum_{\beta=N+1}^{2N-1} |\lambda|^{\beta-1} \sum_{\substack{1 \leq l \leq k \leq N \\ k+l=\beta}} 4^{k-1} C_k^{k-l} \|\varphi\|_{\mathcal{E}}^{k+l} (C_q T)^l (C_q M T)^{k-1} \left(2 + |\lambda| C_q T \|\varphi\|_{\mathcal{E}}\right)^{k-l} \quad (3.14)$$

Let A denotes the quantity $A := |\lambda| C_q T \|\varphi\|_{\mathcal{E}}$ then using this notation (3.14) writes

$$|\Delta_N| \leq \|\psi\|_{\mathcal{E}^*} \|\varphi\|_{\mathcal{E}} \sum_{\substack{1 \leq l \leq k \leq N \\ N+1 \leq k+l \leq 2N-1}} C_k^{k-l} (4MA)^{k-1} A^l \, (2+A)^{k-l}$$

But for all $(k,l) \in (\mathbb{N}^*)^2$ such that $1 \leq l \leq k \leq N$ and $N+1 \leq k+l \leq 2N-1$ we have $k \geq [N/2]$, so we have

$$|\Delta_N| \leq \|\psi\|_{\mathcal{E}^*} \|\varphi\|_{\mathcal{E}} \sum_{k=[N/2]}^{N} (4MA)^{k-1} \, (2+2A)^{k-1}$$

But since (2.2) is satisfied we get $8MA(1+A) < 1$ hence the last inequality shows that Δ_N tends to 0 when N tends to infinity which completes the proof of theorem 2.1.

A Appendix: Planar Binary Trees

Here we will prove the lemma 3.2.1. Let begin with the first part of the lemma which is equivalent to

Lemma A.0.2 *Let b belong to $T(2)$, $\|b\| = \beta$ ($\beta \geq 2$), then we have*

$$\sum_{\substack{0 \leq l \leq k \leq \beta \\ k+l=\beta}} \sum_{\substack{a \in T(2), \|a\|=k \\ E \in \{\varepsilon, Y\}^k | n_Y(E)=l \\ \text{such that } E \propto a = b}} (-1)^{|a|} = 0 \tag{A.1}$$

Proof: Let $b \in T(2)$ be such that $\|b\| = \beta$ ($\beta \geq 2$). Then let us denote by L the integer defined by

$$L := \max\{i \in \mathbb{N} \text{ such that}$$
$$\exists a \in T(2), \|a\| = \beta - i, \ \exists E \in \{\varepsilon, Y\}^{\beta - i} \text{ such that } n_Y(E) = i \text{ and } E \propto a = b\}$$

Then since $\beta \geq 2$ we have $L \geq 1$. Define K by $K := \beta - L$ and let $A \in T(2)$, $\|A\| = K$ and $\hat{E} \in \{\varepsilon, Y\}^K$ such that $\hat{E} \propto A = b$ (and then necessarily $n_Y(\hat{E}) = L$). Note that A is actually unique: it is obtained by removing all pairs of b which are sons of the same vertex. Let denote by I the set of indices $1 \leq i \leq K$ such that $E_i = Y$. Then for all $J \subset I$ we will denote by E^J the K-uplet $E^J := (E_1^J, \dots, E_K^J)$ where for all j in $[\![1, K]\!]$, E_j^J defined by

$$E_j^J = \begin{cases} Y & \text{if } j \in J \\ \varepsilon & \text{if } j \in [\![1, K]\!] \setminus J \end{cases}$$

$\forall j \in J$, $E_j^J = Y$ and $\forall i \in [\![1, K]\!] \setminus J$ $E_i^J = \varepsilon$.

Let k, l be some integers such that $1 \leq l \leq k \leq \beta$ and $k + l = \beta$. Then for all $a \in T_k$ such that there exists $E_a \in \{\varepsilon, Y\}^k$ which satisfies $b = E \propto a$, there is an unique subset $J \subset I$ such that $a = E^J \propto A$ and then we have $k \geq |J| = l \geq 1$. In the other hand for

all $J \subset I$ such that $|J| \geq 1$ there exists an unique $\tilde{E} \in \{\varepsilon, Y\}^{K+|J|}$, $n_Y(\tilde{E}) \geq 1$ such that $\tilde{E} \propto (E^J \propto A) = b$. Hence we have

$$-\sum_{\substack{0 \leq l \leq k \leq \beta \\ k+l=\beta}} \sum_{\substack{a \in T(2), \|a\|=k \\ E \in \{\varepsilon, Y\}^k | n_Y(E)=l \\ \text{such that } E \propto a=b}} (-1)^{|a|} = 0 - \sum_{\substack{J \subset I \\ |J| \leq L}} (-1)^{|E^J \propto A|}$$

but $|E^J \propto A| = K + |J| - 1$ so the previous equality leads to

$$-\sum_{\substack{0 \leq l \leq k \leq \beta \\ k+l=\beta}} \sum_{\substack{a \in T(2), \|a\|=k \\ E \in \{\varepsilon, Y\}^k | n_Y(E)=l \\ \text{such that } E \propto a=b}} (-1)^{|a|} = -\sum_{l=0}^{L} C_L^l (-1)^{K+l-1} = (-1)^K (1-1)^L = 0$$

which completes the proof. ■

Let focus on the second part of lemma 3.2.1

Lemma A.0.3 *Let $p \in \mathbb{N}^*$, $a \in T(2)$ be such that $\|a\| = p$ and $E \in \{\varepsilon, Y\}^p$ then we have $\|E \propto a\| = p + n_Y(E)$ and*

$$\left\langle P^{p+n_Y(E)}_{[\![1,p+n_Y(E)]\!]} \Psi(E \propto a), (\varphi, \dots, \varphi) \right\rangle = \left\langle P^p_{I_E} \Psi(a), (\varphi^{\alpha_1^{I_E}}, \dots, \varphi^{\alpha_p^{I_E}} \right\rangle \tag{A.2}$$

where $I_E := \{j \in [\![1,p]\!]$ such that $E_j = \varepsilon\}$ and $\alpha_j^{I_E} = 2$ if $j \notin I_E$ and 1 otherwise.

<u>Proof</u>: Let $k \in \mathbb{N}^*$ and U belong to $\mathcal{E}^{*k}$, then for all $K \subset [\![1,k]\!]$, for all $t^{\vee K} \in [0,T]^{k-|K|}$ and for all $f^{\vee K} \in (H^q)^{k-|K|}$ we consider the element $U^{\vee K}(t^{\vee K}, f^{\vee K})$ of $\mathcal{E}^{*|K|}$ defined by $\forall \tau \in [0,T]^{|K|}$ and $\forall g \in (H^q)^{|K|}$, $\langle U^{\vee K}(t^{\vee K}, f^{\vee K})(\tau), g \rangle := \left\langle U(\tilde{t}), \tilde{f} \right\rangle$ where $\tilde{t}$ and $\tilde{f}$ are defined by

$$\left\{ \begin{array}{l} \tilde{t}_r := t^{\vee K}_{v(r)} \text{ if } r \notin K \\ \tilde{t}_r := \tau_{k(r)} \text{ if } r \in K \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \tilde{f}_r := f^{\vee K}_{v(r)} \text{ if } r \notin K \\ \tilde{f}_r := g_{k(r)} \text{ if } r \in K \end{array} \right. \tag{A.3}$$

here $v(r) := \mathrm{card}\{k \leq r$ such that $k \notin k\}$ and $k(r) := \mathrm{card}\{k \leq r$ such that $k \in K\}$.

First we will treat the case $n_Y(E) = 1$ then we will see how to generalize the result. For $j \in [\![1,k]\!]$ we define $E^{(j,k)} = (E_1^{(j,k)}, \dots, E_k^{(j,k)}) \in \{\varepsilon, Y\}^k$ by $E_r^{(j,k)} = \varepsilon$ if $r \neq j$ and $E_j^{(j,k)} = Y$. Let $t \in [0,T]^{k-1}$ and $(f_1, \dots, f_{k-1}) \in (H^q)^k$ then we consider the element $\Psi(a)^{\vee\{j\}}(t,f)$ of $\mathcal{E}^*$. In view of the definition of $\Psi(b) = \overline{\Delta}(b)\psi$ we have $\Psi(E^{(j,k)} \propto a) = \overline{\Delta}\left[\Psi(a)^{\vee\{j\}}(t,f)\right] \in \mathcal{E}^{*2}$. Then the calculations done in the proof of proposition 1.2 shows that for all $\varphi \in \mathcal{E}$ solution of (E_λ) we have

$$\left\langle P^2_{[\![1,2]\!]} \overline{\Delta} \left[\Psi(a)^{\vee\{j\}}(t,f)\right], (\varphi, \varphi) \right\rangle = \int_0^T \left\langle \Psi(a)^{\vee j}(t,f)(\tau), \varphi^2(\tau) \right\rangle \mathrm{d}\tau \tag{A.4}$$

Hence using the definition (A.3) of $\Psi(a)^{\vee\{j\}}(t,f)$ we find that the lemma is true if $E = E^{(j,k)}$ i.e. when $n_Y(E) = 1$.

Let $M \in \mathbb{N}^*$ and $E \in \{\varepsilon, Y\}^k$ be such that $n_Y(E) = M$. Then we define $J_E \subset [\![1,k]\!]$ as the set of indices $j \in [\![1,k]\!]$ such that $E_j = Y$, then since $n_Y(E) = M$ we have $|J_E| = M$. We denote $J_E := \{j_1, \ldots, j_M\}$ where $j_M < j_{M-1} < \cdots < j_1$. Then one can show easily that we have

$$b := E \propto a = E^{(j_M, k+M-1)} \propto (E^{(j_{M-1}, k+M-2)} \propto (\cdots \propto (E^{(j_1,k)} \propto a)) \cdots)$$

Hence if we denote by a_1 the Planar Binary Tree $a_1 := E^{(j_{M-1}, k+M-2)} \propto (\cdots \propto (E^{(j_1,k)} \propto a)) \cdots)$ we have $b = E \propto a = E^{(j_M, k+M-1)} \propto a_1$. Then for all

$$\begin{aligned} t^{\{j_M\}} &= (t_1, \ldots, t_{j_M - 1}, t_{j_M+1}, \ldots, t_{k+M-1}) \in [0,T]^{k+M-2} \\ f^{\{j_M\}} &= (f_1, \ldots, f_{j_M - 1}, f_{j_M+1}, \ldots, f_{k+M-1}) \in (H^q)^{k+M-2} \end{aligned}$$

we can use (A.4) and the fact that

$$\overline{\Delta}\left[\Psi(a_1)^{\vee j_M}(t^{\{j_M\}}, f^{\{j_M\}})\right] = \Psi(b)^{\vee\{j_M, j_M+1\}}(t^{\{j_M\}}, f^{\{j_M\}})$$

in order to obtain

$$\begin{aligned} &\left\langle P^2_{[\![1,2]\!]} \Psi(b)^{\vee\{j_M, j_M+1\}}(t^{\{j_M\}}, f^{\{j_M\}}), (\varphi, \varphi)\right\rangle = \\ &\qquad\qquad \int_0^T \left\langle \Psi(a_1)(t), (f_1, \ldots, f_{j_M-1}, \varphi^2(t_{j_M}), f_{j_M+1}, \ldots, f_{k+M-1})\right\rangle \mathrm{d}t_{j_M} \end{aligned}$$

where t denotes the $(k+M-1)$–uplet $t := (t_1, \ldots, t_{j_M-1}, t_{j_M}, t_{j_M+1}, \ldots, t_{k+M-1})$. Then writing $a_1 = E^{(j_{M-1}, k+M-2)} \propto a_2$ one can use the same arguments to show that

$$\begin{aligned} &\left\langle P^4_{[\![1,4]\!]} \Psi(b)^{\vee\{j_M, j_M+1, j_{M-1}+1, j_{M-1}+2\}}(t^{\{j_M, j_{M-1}\}}, f^{\{j_M, j_{M-1}\}}), (\varphi, \varphi, \varphi, \varphi)\right\rangle = \\ &\iint_{[0,T]} \mathrm{d}t_{j_M} \mathrm{d}t_{j_{M-1}} \left\langle \Psi(a_2)(t), (f_1, \ldots, \varphi^2(t_{j_M}), f_{j_M+1}, \ldots, f_{j_{M-1}-1}, \varphi^2(t_{j_{M-1}}), \ldots, f_{k+M-1})\right\rangle \end{aligned}$$

Hence Doing this operation successively for $j_{M-2}, \ldots, j_1$ we finally get

$$\left\langle P^{|K|}_{[\![1,|K|]\!]} \Psi(b)^{\vee K}(t^{\vee K}, f^{\vee K}), (\varphi, \ldots, \varphi)\right\rangle = \iint_{[0,T]^M} \mathrm{d}t_{j_M} \cdots \mathrm{d}t_{j_1} \left\langle \Psi(a)(t), (\tilde{g}_1, \cdots, \tilde{g}_k)\right\rangle \tag{A.5}$$

where $K := \bigcup_{r=1}^M \{j_r + M - r, j_r + M - r + 1\}$ and where $\tilde{g}_r := \varphi^2(t_r)$ if $r \in J_E$ and $\tilde{g}_r := f_{v(r)}$ otherwise. Hence, considering the element $\left\langle P^{|K|}_{[\![1,|K|]\!]} \Psi(b)^{\vee K}(\cdot,\cdot), (\varphi, \ldots, \varphi)\right\rangle$ of $\mathcal{E}^{*(k+M-2M)}$ and using (A.5), we finally get

$$\left\langle P^{k+M}_{[\![1,k+M]\!]} \Psi(b), (\varphi, \ldots, \varphi)\right\rangle = \left\langle P^k_{[\![1,k]\!] \setminus J_E} \Psi(a), (\tilde{h}_1, \cdots, \tilde{h}_k)\right\rangle$$

where $\tilde{g}_r := \varphi^2$ if $r \in J_E$ and $\tilde{h}_r := \varphi$ otherwise i.e. we obtain exactly identity (A.2). ■

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